Advanced Algorithms (Fall 2023)

Greedy and Local Search

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Nanjing University

Outline

1. Greedy Algorithms: Maximum-Weight Independent Set in Matroids
   - Recap: Maximum-Weight Spanning Tree Problem
   - Matroids and Maximum-Weight Independent Set in Matroids
2. Greedy Algorithms: Set Cover and Related Problems
   - 2-Approximation Algorithm for Vertex Cover
   - f-Approximation for Set-Cover with Frequency f
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Maximum Coverage
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Submodular Maximization under a Cardinality Constraint
3. Local Search
   - Warmup Problem: 2-Approximation for Maximum-Cut
   - Local Search for Uncapacitated Facility Location Problem
   - Local Search for UFL: Analysis for Connection Cost
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Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$

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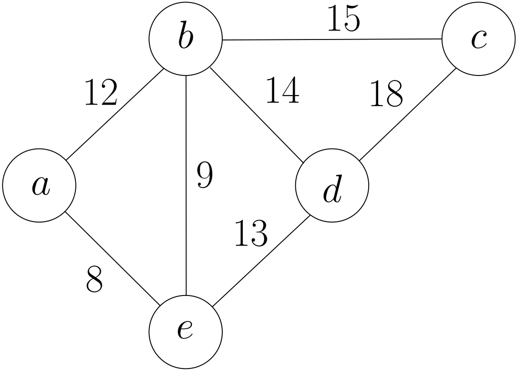

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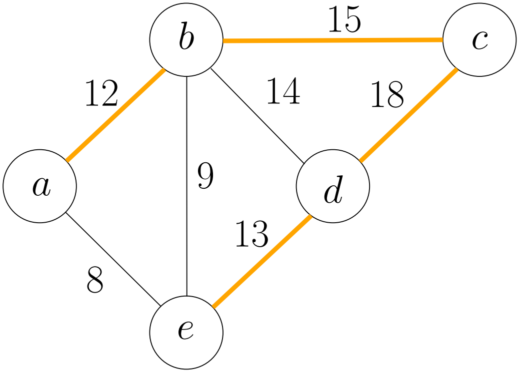

Kruskal's Algorithm for Maximum-Weight Spanning Tree

$F \leftarrow \emptyset$
sort edges in E in non-increasing order of weights w
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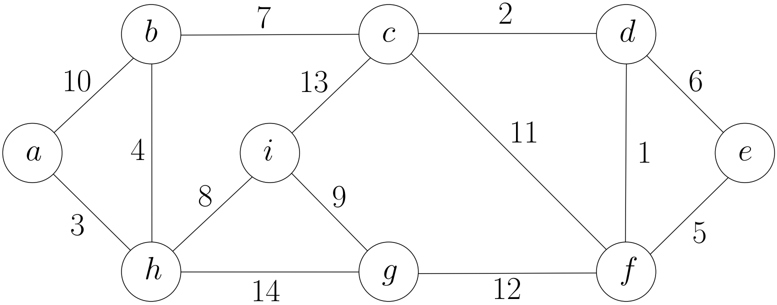

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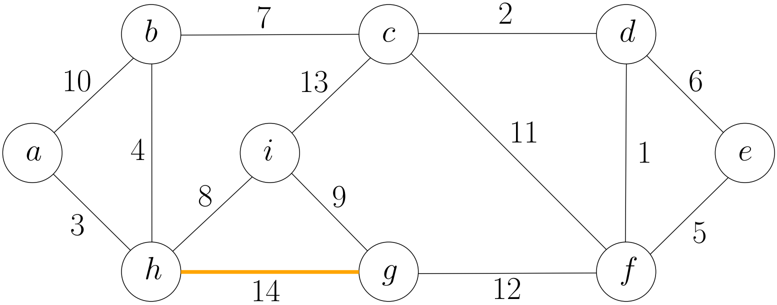

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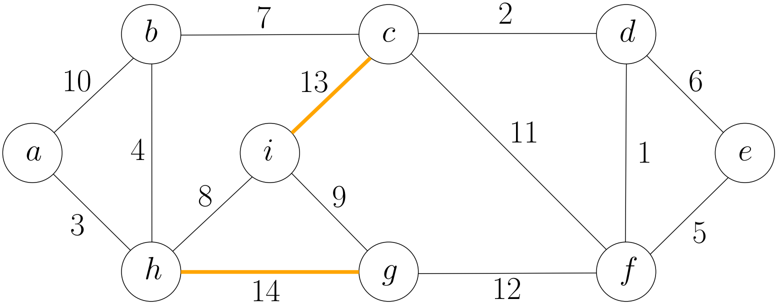

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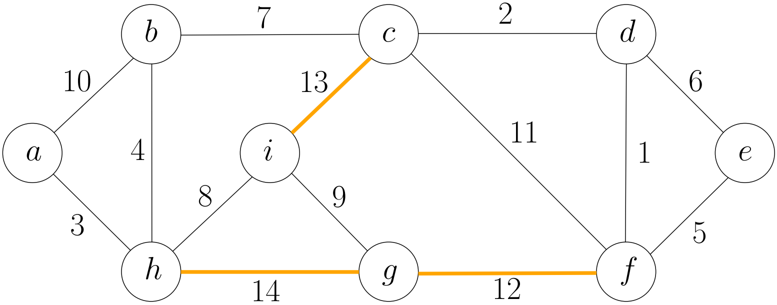

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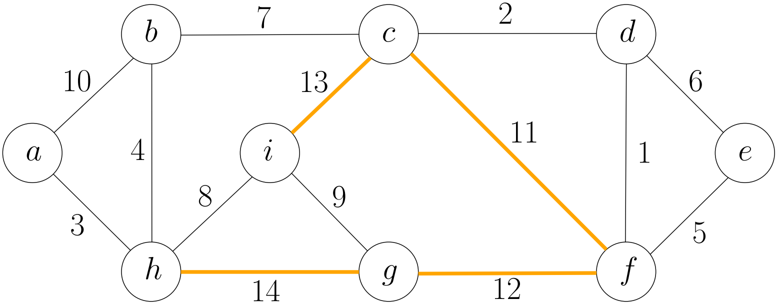

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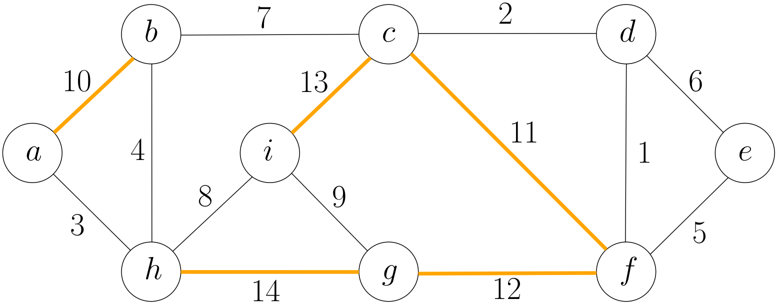

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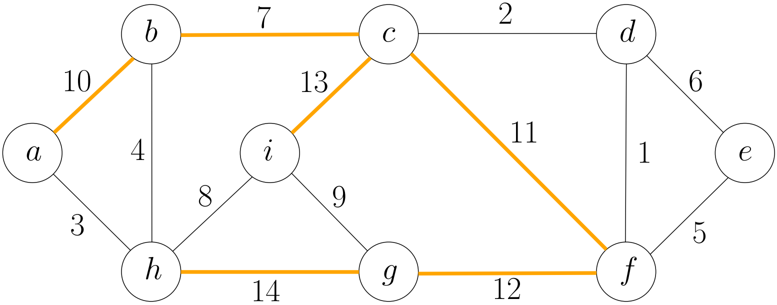

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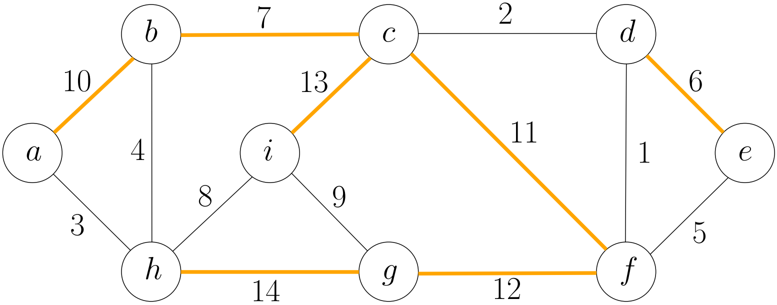

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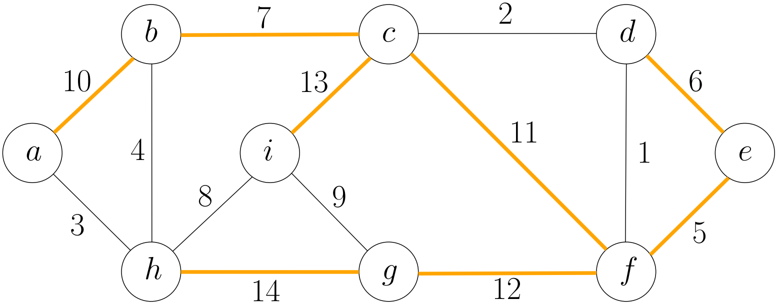

Proof of Correctness of Kruskal's Algorithm

Maximum-Weight Spanning Tree (MST) with Pre-Selected Edges

Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^E$

a set $F_0 \subseteq E$ of edges, that does not contain a cycle

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Lemma (Key Lemma) Given an instance $(G = (V, E), w, F_0)$ of the MST with pre-selected edges problem, let e^* be the maximum weight edge in $E \setminus F_0$ such that $F_0 \cup \{e^*\}$ does not contain a cycle. Then there is an optimum solution $T = (V, E_T)$ to the instance with $e^* \in E_T$.

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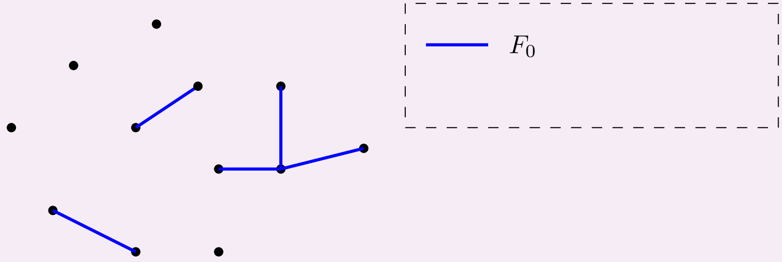
Proof of Key Lemma.



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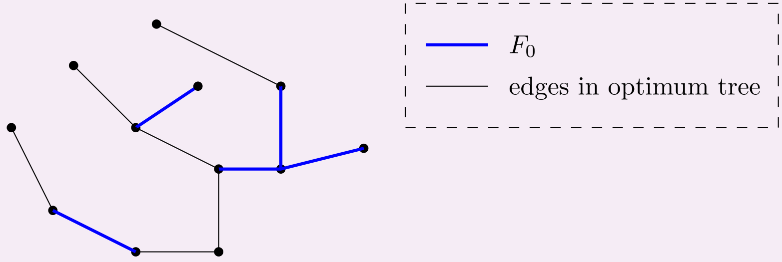

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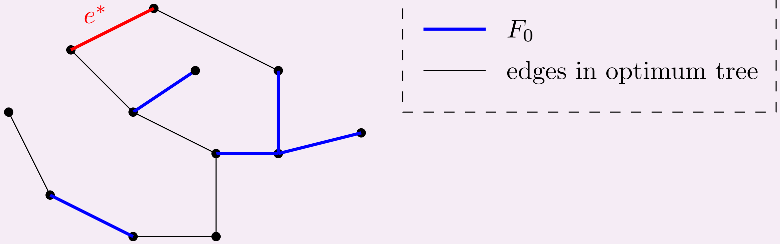

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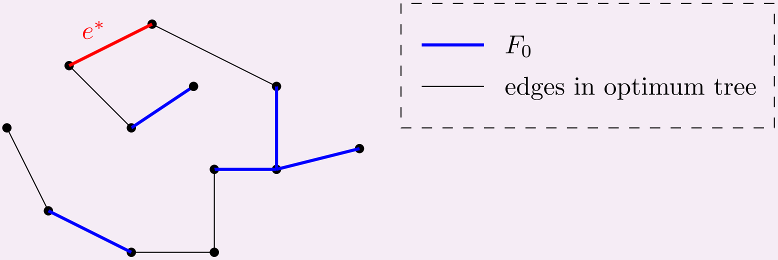

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A General Maximization Problem

Input: E: the ground set of elements

$w \in \mathbb{Z}_{>0}^E$: weight vector on elements

$\mathcal{S}$: an (implicitly given) family of subsets of E

- $\emptyset \in \mathcal{S}$
- $\mathcal{S}$ is downward closed: if $A \in \mathcal{S}, B \subsetneq A$, then $B \in \mathcal{S}$.

Output: $A \in \mathcal{S}$ that maximizes $\sum_{e \in A} w_e$

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- maximum-weight spanning tree: $\mathcal{S} =$ family of forests

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- Knapsack Packing: given elements E, where every element has a value and a cost, and a cost budget C, the goal is to find a maximum value subset of items with cost at most C

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- Matroids: cases where greedy algorithm is optimum

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
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Proof of Exchange Property.

- $|B| < |A| \Rightarrow (V, B)$ has more CC than (V, A).
- Some edge in A connects two different CC of (V, B). □

Feasible Family for Knapsack Packing Does Not Satisfy Augmentation Property

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- $\{1,2\}, \{3\} \in \mathcal{I}$, but $\{1,3\}, \{2,3\} \notin \mathcal{I}$.

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Theorem The greedy algorithm gives optimum solution for the maximum-weight independent set problem in a matroid.

Lemma (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}_{>0}^{E}$, $A \in \mathcal{I}$,
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while $|S'| < |S|$ **do**
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- S' and S differ by exactly one element
- $w(S') := \sum_{e \in S'} w_e \geq w(S) \implies S'$ is also optimum □

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- Partition Matroid: partition $(E_1, E_2, \cdots, E_t)$ of E, positive integers $k_1, k_2, \cdots, k_t$

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- Laminar Matroid: laminar family of subsets of E $\{E_1, E_2, \cdots, E_t\}$, positive integers $k_1, k_2, \cdots, k_t$
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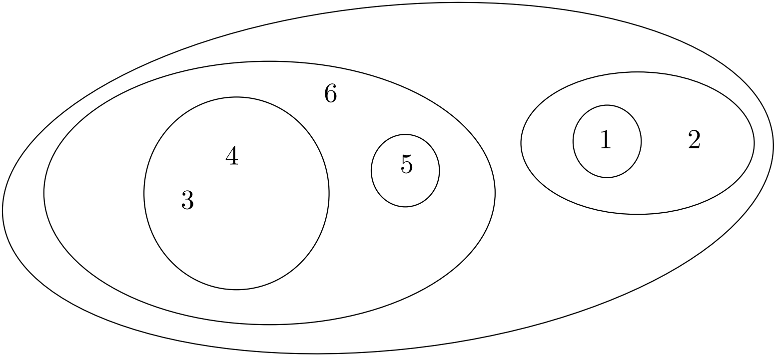
Def. A family $\{E_1, E_2, \cdots, E_t\}$ of subsets of E is said to be laminar if for every two distinct subsets E_i, E_j in the family, we have $E_i \cap E_j = \emptyset$ or $E_i \subsetneq E_j$ or $E_j \subsetneq E_i$.

- $\Big\{\{1\},\{1,2\},\{3,4\},\{5\},\{3,4,5,6\},\{1,2,3,4,5,6\}\Big\}$ is a laminar family.



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- Linear Matroid: a vector $\vec{v}_e \in \mathbb{R}^d$ for every $e \in E$

  $$\mathcal{I} = \{A \subseteq E : \text{vectors } \{\vec{v}_e\}_{e \in A} \text{ are linearly independent}\}$$

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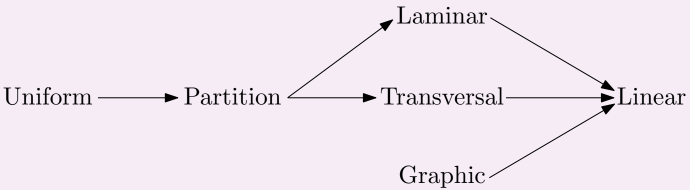
- E: the ground set $\qquad$ $\mathcal{I}$: the family of independent sets
- Graphic Matroid: graph $G = (V, E)$

  $\mathcal{I} = \{A \subseteq E : (V, A) \text{ is a forest}\}$
- Transversal Matroid: a bipartite graph $G = (E \uplus B, \mathcal{E})$

  $\mathcal{I} = \{A \subseteq E : \text{there is a matching in } G \text{ covering } A\}$
- Linear Matroid: a vector $\vec{v}_e \in \mathbb{R}^d$ for every $e \in E$

  $\mathcal{I} = \{A \subseteq E : \text{vectors } \{\vec{v}_e\}_{e \in A} \text{ are linearly independent}\}$

Relationship between matroids



Other Terminologies Related To a Matroid $\mathcal{M} = (E, \mathcal{I})$

- A subset of E that is not independent is dependent.
- A maximal indepent set is called a basis (plural: bases)
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Lemma All bases of a matroid have the same size.

Proof.

By exchange property. □

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Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of a subset A of E, denoted as $r_{\mathcal{M}}(A)$, is defined as the size of the maximum independent subset of A. $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

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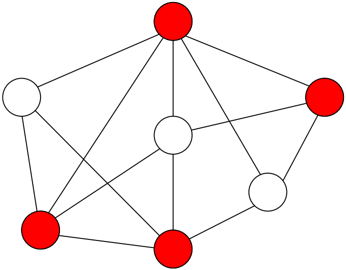
Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



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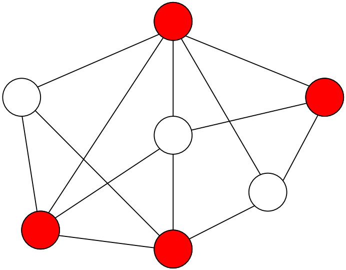

Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

$E' \leftarrow E, C \leftarrow \emptyset$
while $E' \neq \emptyset$ **do**
    let v be the vertex of the maximum degree in (V, E')
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- The logarithmic factor is tight for this algorithm

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$E' \leftarrow E, C \leftarrow \emptyset$
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Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

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every $j \in U$ appears in at most f subsets in $\{S_1, S_2, \cdots, S_n\}$

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Vertex Cover = Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

```
C ← ∅
while ⋃_{i∈C} S_i ≠ U do
    let e be any element in U \ ⋃_{i∈C} S_i
    C ← C ∪ {i ∈ [m] : e ∈ S_i}
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f-Approximation Algorithm for Set Cover with Frequency f

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3: let e be any element in $U \setminus \bigcup_{i \in C} S_i$
4: $C \leftarrow C \cup \{i \in [m] : e \in S_i\}$
5: **return** C

Theorem The algorithm is a f-approximation algorithm.

Proof.

- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ □

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- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

$$u_t \leq \left(1 - \frac{1}{g}\right) \cdot u_{t-1}.$$

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- $S_1^* \cup S_2^* \cup \cdots \cup S_g^* = U$
- at beginning of step t, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. □

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \leq \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
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$(1 - c) \ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1 - c) \ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P = NP. [Dinur, Steuer 2014]

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- The $(1-\frac{1}{e})$-approximation extends to a more general problem.

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Def. Let $n \in \mathbb{Z}_{>0}$. A set function $f : 2^{[n]} \to \mathbb{R}$ is called submodular if it satisfies one of the following three equivalent conditions:

(1) $\forall A, B \subseteq [n]$:

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

(2) $\forall A \subseteq B \subsetneq [n], i \in [n] \setminus B$:

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A).$$

(3) $\forall A \subseteq [n], i, j \in [n] \setminus A, i \neq j$:

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(2) $\forall A \subseteq B \subsetneq [n], i \in [n] \setminus B$:

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- (2): diminishing marginal values: the marginal value by getting i when I have B is at most that when I have $A \subseteq B$.

Def. Let $n \in \mathbb{Z}_{>0}$. A set function $f : 2^{[n]} \to \mathbb{R}$ is called submodular if it satisfies one of the following three equivalent conditions:

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- $(1) \Rightarrow (2) \Rightarrow (3)$, $\qquad (3) \Rightarrow (2) \Rightarrow (1)$

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$(1-\frac{1}{e})$-Approximation for Submodular Maximization with Cardinality Constraint

Submodular Maximization under a Cardinality Constraint

Input: An oracle to a non-negative monotone submodular function $f: 2^{[n]} \to \mathbb{R}_{\geq 0}$, $k \in [n]$

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Greedy Algorithm for the Problem

1: $S \leftarrow \emptyset$
2: **for** $t \leftarrow 1$ to k **do**
3: choose the i that maximizes $f(S \cup \{i\})$
4: $S \leftarrow S \cup \{i\}$
5: **return** S

Theorem Greedy algorithm gives $(1 - \frac{1}{e})$-approximation for submodular-maximization under a cardinality constraint.

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Proof.

- o: optimum value
- p_t: value obtained by greedy algorithm after step t
- need to prove: $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o-p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ □

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Lemma A non-negative submodular set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is sub-additive.

Proof.

For $A, B \subseteq [n]$, we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.
So, $f(A \cup B) \leq f(A) + f(B)$ as $f(A \cap B) \geq 0$. □

Lemma Let $f : 2^{[n]} \to \mathbb{R}$ be submodular. Let $S \subseteq [n]$, and $f_S(A) = f(S \cup A) - f(S)$ for every $A \subseteq [n]$. (f_S is the marginal value function for set S.) Then f_S is also submodular.

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Proof.

- Let $A, B \subseteq [n] \setminus S$; it suffices to consider ground set $[n] \setminus S$.

$$\begin{aligned} & f_S(A \cup B) + f_S(A \cap B) - f_S(A) + f_S(B) \\ &= f(S \cup A \cup B) - f(S) + f(S \cup (A \cap B)) - f(S) \\ &\quad - \Big(f(S \cup A) - f(S) + f(S \cup B) - f(S)\Big) \\ &= f(S \cup A \cup B) + f(S \cup (A \cap B)) - f(S \cup A) - f(S \cup B) \\ &\leq 0 \end{aligned}$$

- The last inequality is by $S \cup A \cup B = (S \cup A) \cup (S \cup B)$, $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$ and submodularity of f. □

Proof of $p_t \geq p_{t-1} + \frac{o - p_{t-1}}{k}$.

- $S^* \subseteq [n]$: optimum set, $|S^*| = k$, $o = f(S^*)$
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$$f(S \cup \{i\}) - f(S) \geq \frac{1}{k}(f(S^*) - f(S))$$

$$p_t \geq f(S \cup \{i\}) \geq p_{t-1} + \frac{1}{k}(o - p_{t-1}) \qquad \square$$

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Local Search for Maximum-Cut

Maximum-Cut

Input: Graph $G = (V, E)$

Output: partition of V into $(S, T = V \setminus S)$ so as to maximize $|E(S,T)|$, $E(S,T) = \{uv \in E : u \in S \wedge v \in T\}$.

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Local-Search for Maximum-Cut

```
1: (S, T) ← any cut
2: while ∃v ∈ V, changing side of v increases cut value do
3:     switch v to the other side in (S, T)
4: return (S, T)
```

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- d_v: degree of v

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- $\forall v \in S : E(v, S) \leq E(v, T) \Rightarrow |E(v, S)| \geq \frac{1}{2} d_v$
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- adding all inequalities:

$$2|E(S, T)| \geq \frac{1}{2} \sum_{v \in V} d_v = |E|.$$

- So $|E(S, T)| \geq \frac{1}{2}|E| \geq \frac{1}{2}$(value of optimum cut). □

- The following algorithm also gives a 2-approximation

Greedy Algorithm for Maximum-Cut

$S \leftarrow \emptyset, T \leftarrow \emptyset$
for every $v \in V$, in arbitrary order **do**
    adding v to S or T so as to maximize $|E(S,T)|$
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- Under Unique-Game-Conjecture (UGC), the ratio is best possible

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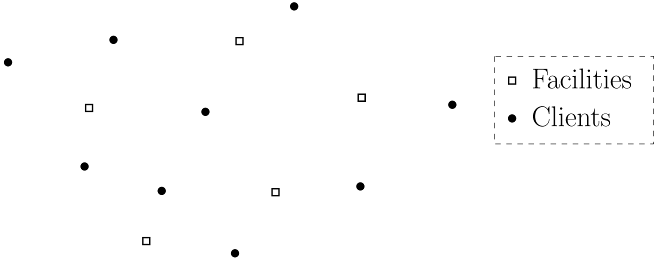
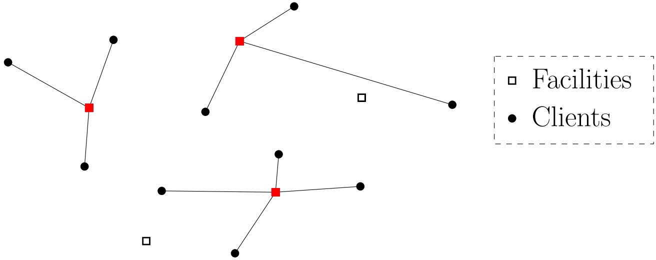


Uncapacitated Facility Location

Input: F: Facilities C: Clients

d: metric over $F \cup C$ $(f_i)_{i \in F}$: facility costs



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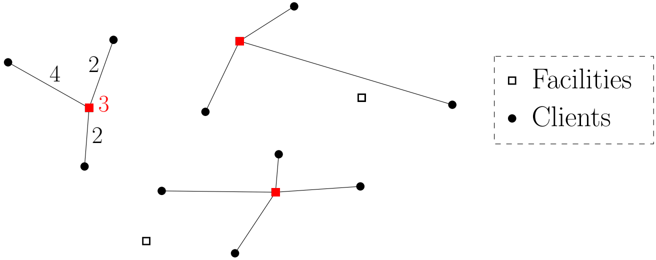
$d(j, S)$: smallest distance between j and a facility in S



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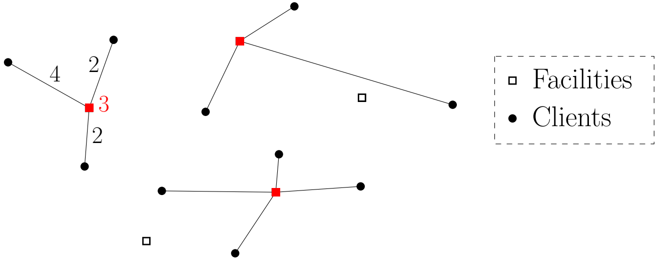

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- Best-approximation ratio: 1.488-Approximation [Li, 2011]
- 1.463-hardness, $1.463 \approx$ root of $x = 1 + 2e^{-x}$

- $\text{cost}(S) := \sum_{i \in S} f_i + \sum_{j \in C} d(j, S), \forall S \subseteq F$

Local Search Algorithm for Uncapacitated Facility Location

$S \leftarrow$ arbitrary set of facilities
while exists $S' \subseteq F$ with $|S \setminus S'| \leq 1$, $|S' \setminus S| \leq 1$ and $\text{cost}(S') < \text{cost}(S)$ **do**
    $S' \leftarrow S$
return S

- The algorithm runs in pseodu-polynomial time, but we ignore the issue for now.

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S is a local optimum, under the following local operations

- add$(i), i \notin S$: $S \leftarrow S \cup \{i\}$
- delete$(i), i \in S$: $S \leftarrow S \setminus \{i\}$
- swap$(i, i'), i \in S, i' \notin S$: $S \leftarrow S \setminus \{i\} \cup \{i'\}$

- S: the local optimum returned by the algorithm
- S^*: the (unknown) optimum solution

$$F := \sum_{i \in S} f_i \qquad\qquad C := \sum_{j \in C} d(j, S)$$

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Lemma (analysis for facility cost) $F \leq F^* + 2C^*$

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So, $F + C \leq 2F^* + 3C^* \leq 3(F^* + C^*)$

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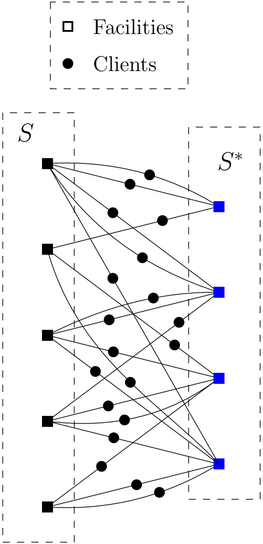
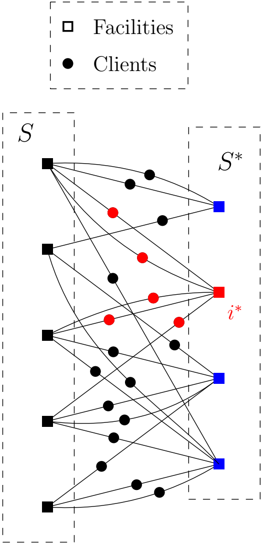


Analysis of C



Analysis of C

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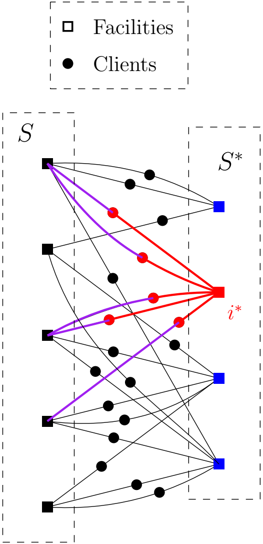
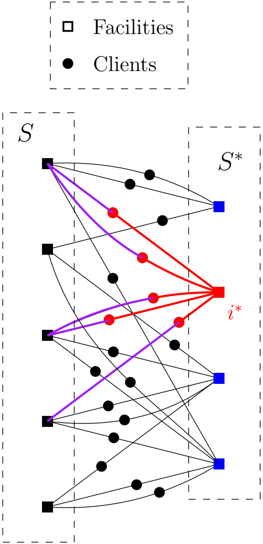


Analysis of C

- adding i^* does not increase the cost:



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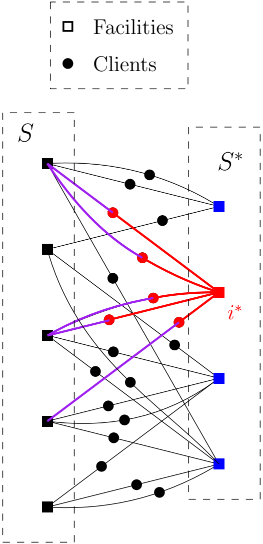
$$\sum_{j \in \sigma^{*-1}(i^*)} c_{\sigma(j)j} \leq f_{i^*} + \sum_{j \in \sigma^{*-1}(i^*)} c_{i^*j}$$



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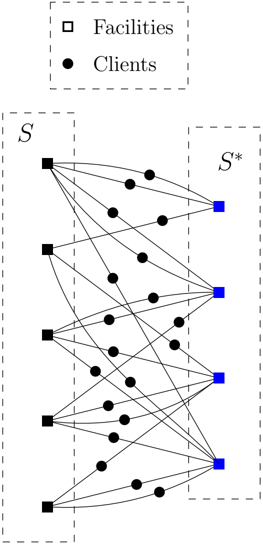
- summing up over all $i^* \in F^*$, we get

$$\sum_{j \in J} d(\sigma(j), j) \le \sum_{i^* \in F^*} f_{i^*} + \sum_{j \in J} d(\sigma^*(j), j)$$

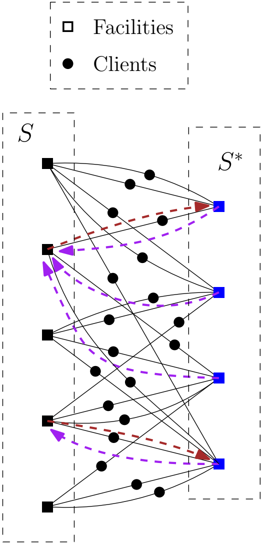
$$C \le F^* + C^*$$

Outline

1. Greedy Algorithms: Maximum-Weight Independent Set in Matroids
   - Recap: Maximum-Weight Spanning Tree Problem
   - Matroids and Maximum-Weight Independent Set in Matroids
2. Greedy Algorithms: Set Cover and Related Problems
   - 2-Approximation Algorithm for Vertex Cover
   - f-Approximation for Set-Cover with Frequency f
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Maximum Coverage
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Submodular Maximization under a Cardinality Constraint
3. Local Search
   - Warmup Problem: 2-Approximation for Maximum-Cut
   - Local Search for Uncapacitated Facility Location Problem
   - Local Search for UFL: Analysis for Connection Cost
   - Local Search for UFL: Analysis for Facility Cost



Analysis of F



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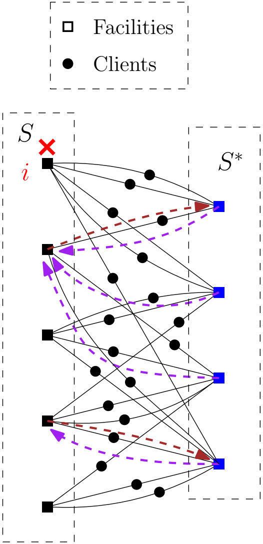

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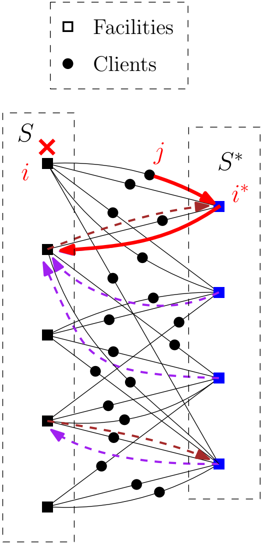

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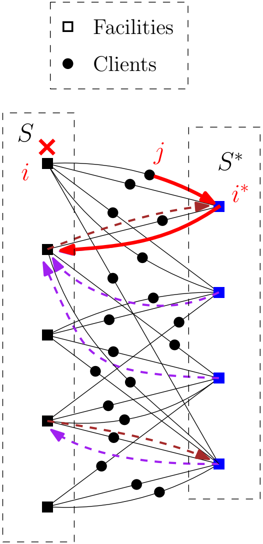

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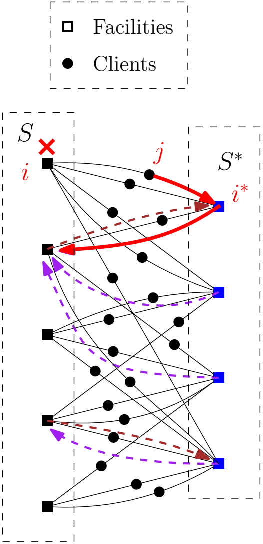
  - reconnection distance is at most

$$c_{i^*j} + c_{i^*\phi(i^*)} \leq c_{i^*j} + c_{i^*i}$$
$$\leq c_{i^*j} + c_{i^*j} + c_{ij} = 2c_{i^*j} + c_{ij}$$



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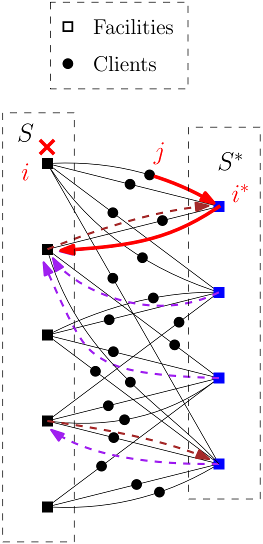

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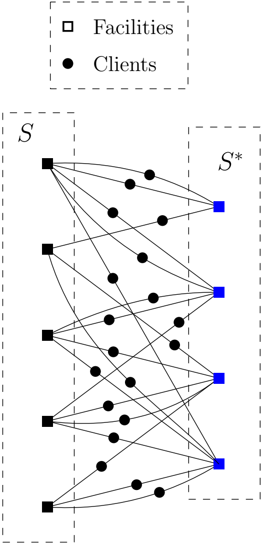

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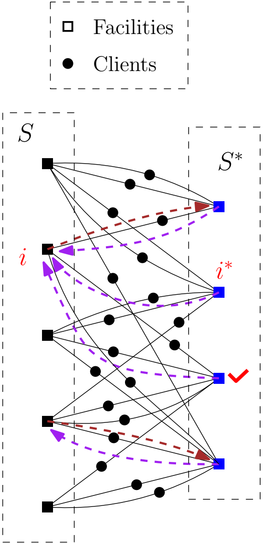

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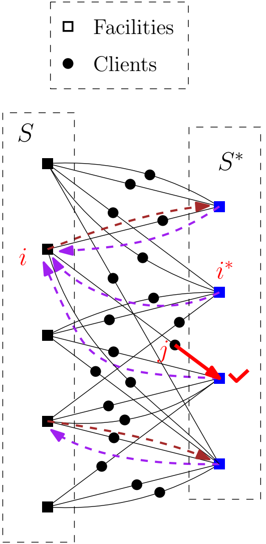

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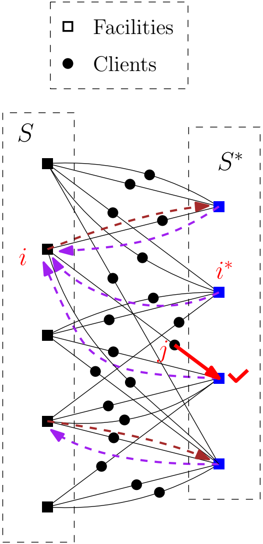

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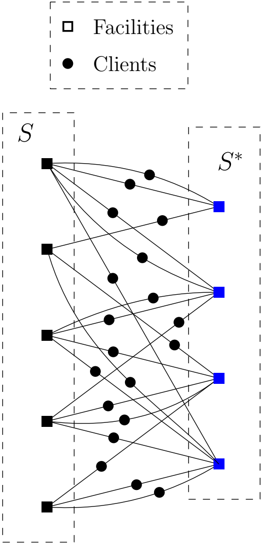

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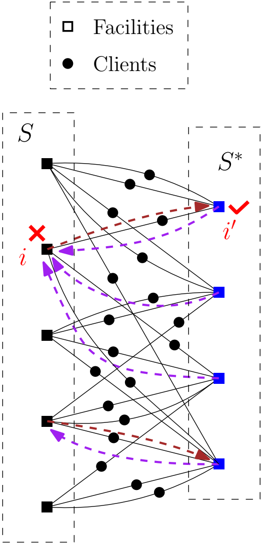

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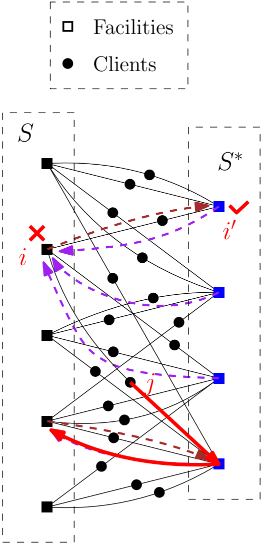
- $i \in S, \phi^{-1}(i) \neq \emptyset, \phi(i') = i, \psi(i) = i'$: consider swap$(i, i')$



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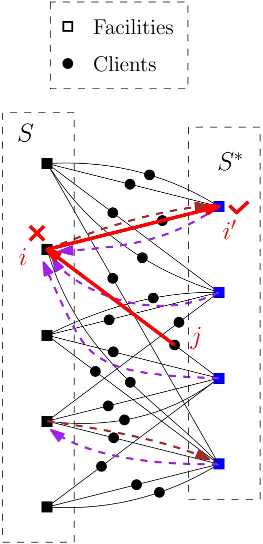

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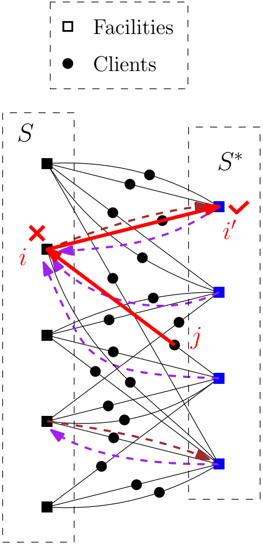

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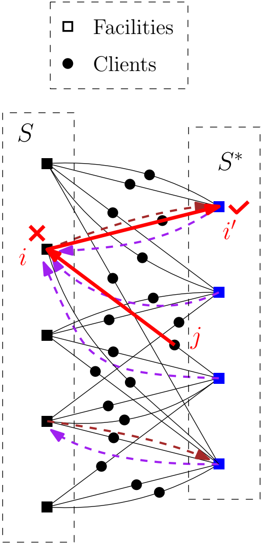
    $$c_{ij} + c_{ii'} - c_{ij} = c_{ii'} \leq c_{i\sigma^*(j)} \leq c_{ij} + c_{\sigma^*(j)j}$$



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  $$c_{ij} + c_{ii'} - c_{ij} = c_{ii'} \leq c_{i\sigma^*(j)} \leq c_{ij} + c_{\sigma^*(j)j}$$
  - $$f_i \leq f_{i'} + 2 \sum_{j \in \sigma^{-1}(i): \phi(\sigma^*(j)) \neq i} c_{\sigma^*(j)j} + \sum_{j \in \sigma^{-1}(i): \phi(\sigma^*(j)) = i} (c_{ij} + c_{\sigma^*(j)j})$$

- $i \in S$ is not paired: $f_i \leq 2 \sum_{j \in \sigma^{-1}(i)} c_{\sigma^*(j)j}$
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- $i \in S$ and $i' \in S^*$ are paired:

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- summing all the inequalities:

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$$\Rightarrow \qquad F + C \leq 2F^* + 3C^* \leq 3(F^* + C^*)$$

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Exercise: scaling facility costs by some $\lambda > 1$ can give a $(1 + \sqrt{2})$-approximation.

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Exercise: scaling facility costs by some $\lambda > 1$ can give a $(1+\sqrt{2})$-approximation.

- Handling pseudo-polynomial running time issue:

Local Search Algorithm for Uncapacitated Facility Location

```
S ← arbitrary set of facilities, δ ← ε/(4|F|)
while exists S' ⊆ F with |S \ S'| ≤ 1, |S' \ S| ≤ 1 and
    cost(S') < (1 − δ)cost(S) do
        S' ← S
return S
```